MATH 307

2.5: Autonomous Equations

Any ODE of the form $\frac{dy}{dt} = \underbrace{f(y)}_{\text{No } t\text{'s}}$ is called autonomous.

All autonomous equations are separable $\boxed{\int \frac{1}{f(y)}\,dy = \int dt = t + c}$

Qualitative Analysis

Recall: The equilibrium sol'ns to $\frac{dy}{dt} = f(t,y)$ are all constant sol'n $y(t) = c$ for which $f(t,c) \overset{?}{=} 0$

Ex) · $\frac{dy}{dt} = -0.1(y - 70) \overset{?}{=} 0$

$\Rightarrow y(t) = 70$ only equilibrium sol'n.

· $\frac{dy}{dt} = (y-20)^2(10-y)(e^y - 1) \overset{?}{=} 0$

$\Rightarrow y(t) = 0,\ y(t) = 10,\ y(t) = 20.$

Assuming $f(t,y)$ and $\frac{\partial f}{\partial y}(t,y)$ are continuous, other sol'ns can't intersect an equilibrium. So you tend to get asymptotic behavior

Def'n) An equilibrium sol'n $y(t) = c$ is

- asymptotically STABLE if sol'ns nearby approach c as $t \to \infty$.
- " UNSTABLE if sol'ns nearby don't approach c as $t \to \infty$
- " SEMISTABLE if approaches on one side but not the other.

SHOW PICS!

To classify:

① FIND EQUILIBRIUM

② STUDY/GRAPH $f(y)$

(a) MAKE A VERTICAL PHASE LINE, MAKE TICKS AT EQ.

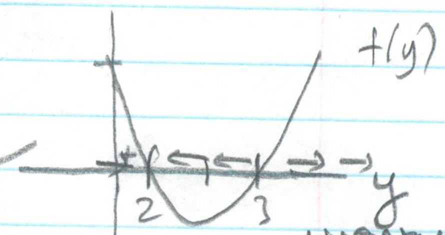
(b) DETERMINE IF $f(y)$ IS POS OR NEG BETWEEN TICKS

③ MAKE CONCLUSIONS!

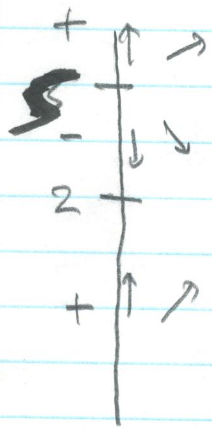
Ex) $\frac{dy}{dt} = y^2 - 6y + 5 = f(y)$

$f(y) = (y-2)(y-3) \overset{?}{=} 0$

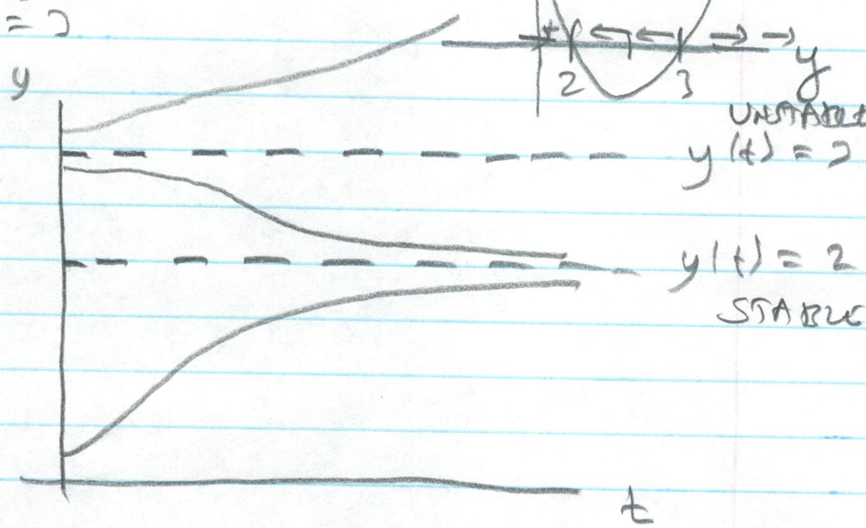
$y = 2,\ y = 3$



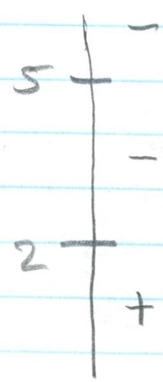
$\frac{dy}{dt} = f(y)$

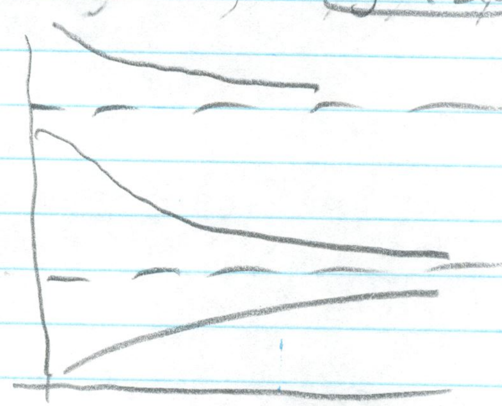


PHASE LINE



Ex) $\frac{dy}{dt} = (2-y)(y-5)^2$ ... $\boxed{\text{you do}}$ ...





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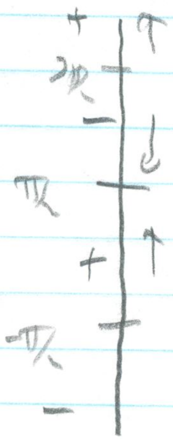
Ex) $\frac{dy}{dt} = \cos(y)$ $\qquad$ $y =, y = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$



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Population models

$y(t)$ = SIZE OF POPULATION AT TIME t

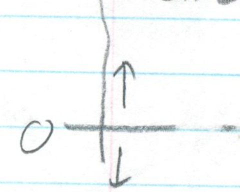
$\frac{dy}{dt}$ = RATE OF CHANGE (PEOPLE ADDED/SUBTRACT PER YEAR)

NATURAL UNRESTRICTED GROWTH Ideal conditions.

$$\boxed{\frac{dy}{dt} = ry, \quad y(0) = y_0} \qquad r = \text{relative growth rate}$$

$$\boxed{y = y_0 e^{rt}}$$ EXPONENTIAL GROWTH!

PHASE LINE



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RESTRICTED GROWTH restricted growth

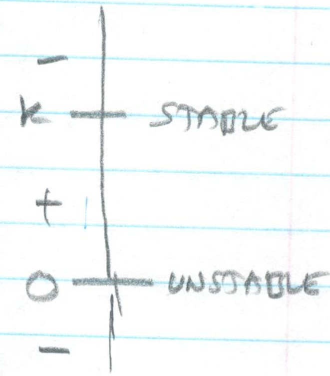
$$\frac{dy}{dt} = h(y)\,y \quad \begin{cases} h(y) \approx r & \text{for } y \text{ "SMALL"} \\ h(y) \to 0 & \text{as } y \to \text{SOME CAPACITY / SATURATION LEVEL} \end{cases}$$

Logistic $\boxed{\dfrac{dy}{dt} = r\left(1 - \frac{1}{k}y\right)y}$ $y(0) = y_0$

EQUILIBRIUM? $y(t) = 0$, $y(t) = k$

$$\boxed{y(t) = \frac{y_0 k}{y_0 + (k - y_0)e^{-kt}}}$$

$t \to \infty \Rightarrow y(t) \to k$

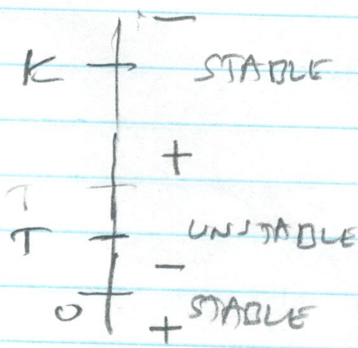


LOGISTIC WITH THRESHOLD

$$\frac{dy}{dt} = -r\left(1 - \frac{1}{T}y\right)\left(1 - \frac{1}{k}y\right)y$$

$0 < T < k$

$y(t) = 0$, $y(t) = T$, $y(t) = k$



BEHAVIOR?

PREVIEW EXACT, DO SOME IMPLICIT DIFFERENTIATION